

Energy Inequalities in Quantum Field Theory*

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Abstract

Quantum fields are known to violate all the pointwise energy conditions of classical general relativity. We review the subject of quantum energy inequalities: lower bounds satisfied by weighted averages of the stress-energy tensor, which may be regarded as the vestiges of the classical energy conditions after quantisation. Contact is also made with thermodynamics and related issues in quantum mechanics, where such inequalities find analogues in sharp Gårding inequalities.

1 Introduction: Energy conditions in General Relativity

In classical relativity, the energy-momentum current density seen by an observer with four-velocity v^b is defined to be $\Pi^a = T^a_b v^b$, where T_{ab} is the stress-energy tensor of surrounding matter.¹ The requirement that Π^a should be timelike and future-directed is known as the **dominant energy condition** (DEC) and is a natural expression of the fundamental relativistic principle that no influence may propagate faster than light. This interpretation is borne out by the fact that a conserved stress-energy tensor which obeys the DEC will vanish on the domain of dependence of any closed achronal set on which it vanishes (see Sec. 4.3 in [1]), so the DEC prohibits acausal propagation of stress-energy. The DEC may, equivalently, be formulated as the requirement that

$$T_{ab} u^a v^b \geq 0 \tag{1}$$

for all timelike, future-directed u^a, v^b ; it also contains (as the special case $u^a = v^b$) the **weak energy condition** (WEC), the assertion that all timelike observers measure positive energy density. By continuity, this implies the **null energy condition** (NEC), namely that $T_{ab} k^a k^b \geq 0$ for all null k^a .

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¹Our metric signature is $+- - -$; units with $\hbar = c = 1$ will also be adopted.

The classical energy conditions are satisfied by most classical matter models and have several important consequences. Matter obeying the NEC tends to focus null geodesic congruences, a fact which plays a key role in the singularity theorems [1], and the WEC (respectively, DEC) is a sufficient condition for the positivity of the ADM (respectively, Bondi) mass [2, 3]. However, quantum fields have long been known to violate all such pointwise energy conditions [4] and, in many models, the energy density is in fact unbounded from below on the class of physically reasonable states. Moreover, the existence of negative energy densities draws indirect experimental support from the Casimir effect [5]. In this contribution we review these phenomena and the extent to which quantum fields satisfy weaker energy conditions, which may be called **quantum energy inequalities** (QEIs). We also describe connections between such inequalities and thermodynamical stability, and some wider parallels in quantum mechanics. Finally, the physical picture of energy condition violation which emerges from these results is briefly discussed.

2 The existence of negative energy densities in quantum field theory

In 1965, Epstein, Glaser and Jaffe proved that the energy density in any Wightman field theory necessarily admits negative expectation values (unless it is trivial) [4]. Here, we give an elementary argument for this conclusion, the basis of which goes back at least to [6], and which applies quite generally.

Consider a theory specified by a Hilbert space \mathcal{H} , a dense domain $D \subset \mathcal{H}$ and a distinguished vector $\Omega \in \mathcal{H}$, which we call the vacuum. In this context, a field is an operator valued distribution on spacetime with the property that $T(f)D \subset D$ for all test functions f . In addition, we assume only that T enjoys the Reeh–Schlieder property that no $T(f)$ can annihilate the vacuum (for nontrivial f) and, for simplicity, that $T(f)$ has vanishing vacuum expectation values, which corresponds to adopting the vacuum as the zero of energy. This is what one would expect of the energy density in Minkowski space; one may easily adapt the argument to cope with nonvanishing vacuum expectation values by treating $\tilde{T}(f) = T(f) - \langle \Omega | T(f) | \Omega \rangle \mathbf{1}$ in place of $T(f)$. With these assumptions in place, let f be any nonnegative test function and define (for $\alpha \in \mathbb{R}$)

$$\psi_\alpha = \cos \alpha \Omega + \sin \alpha \frac{T(f)\Omega}{\|T(f)\Omega\|}. \quad (2)$$

Then an elementary calculation yields

$$\langle \psi_\alpha | T(f) \psi_\alpha \rangle = \zeta \sin 2\alpha + \eta(1 - \cos 2\alpha), \quad (3)$$

where

$$\zeta = \|T(f)\Omega\| \quad \text{and} \quad \eta = \frac{\langle \Omega | T(f)^3 \Omega \rangle}{2\|T(f)\Omega\|^2}. \quad (4)$$

By minimising over α , we therefore find

$$\inf_{\substack{\psi \in D \\ \|\psi\|=1}} \langle \psi | T(f) \psi \rangle \leq \eta - \sqrt{\eta^2 + \zeta^2}, \quad (5)$$

which is negative. Of course, this argument has very little to do with quantum field theory and almost nothing to do with energy density *per se*: the key ingredient is the linear structure of Hilbert space, and similar arguments also apply in quantum mechanics.

We may pursue this line of reasoning a little further if we may assume that the vacuum admits a nontrivial scaling limit for T with positive canonical dimension (see [7] and Sec. VII.3.2 of [8]) and with a nontrivial two-point function in the limit. As shown in the Appendix, one may then choose a sequence f_n of nonnegative test functions tending to a δ -function so that $\zeta_n \rightarrow \infty$, while η_n/ζ_n tends to a finite limit. It then follows from Eq. (5) that the expectation value of T at a point (if it exists) is unbounded from below as the state varies in D .

3 Quantum Energy Inequalities

Although one cannot expect reasonable quantum field theories to satisfy any of the point-wise classical energy conditions, one may still hope that there would be some vestige of these conditions in quantum field theory: after all, they ought to emerge from the quantum field theory in the classical limit. This leads to the conjecture that *smear*ed energy densities might satisfy state-independent bounds, which become progressively weaker as the support of the smearing function shrinks, and tighter as it grows. Bounds of this type, known as **Quantum Weak Energy Inequalities**² (QWEIs) were first proved by Ford [11] who was initially guided by thermodynamic considerations [12] (see also Sec. 4). The original bound actually concerned the energy flux, but was soon adapted to the energy density of the scalar and electromagnetic fields in Minkowski space [13, 14]. In these bounds, the energy density is averaged along an inertial trajectory against a Lorentzian weight; for example, the massless scalar field in four-dimensions was shown to obey

$$\int dt \frac{\tau \langle T_{00}(t, \mathbf{x}) \rangle_\psi}{\pi(t^2 + \tau^2)} \geq -\frac{3}{32\pi^2\tau^4} \quad (6)$$

for a large class of states ψ . The parameter τ sets the timescale over which the average is taken; as hoped, we find that the bound is tighter as τ increases (leading to a proof of the averaged weak energy condition (AWEC) in the limit $\tau \rightarrow \infty$). The fact that the bound diverges as $\tau \rightarrow 0$ is consistent with the unboundedness from below of the energy density at a point. Eq. (6) is of course reminiscent of the time–energy uncertainty relation (although this is not an ingredient of the proof). Bounds of this type were generalised to ultrastatic spacetimes by Pfenning and Ford [15], for averages along static trajectories with the Lorentzian weight. In curved spacetimes (or even in compact flat spacetimes) it is of course possible to have a constant negative renormalised energy density, which could not satisfy a bound of the form above. The quantity appearing in the results of [15] is, instead, the difference between the renormalised energy density in state ψ and that taken in the vacuum, which we might refer to as the normal ordered energy density. Thus

²The original terminology was simply “quantum inequality” (QI); the more specific term QWEI was introduced later [9], as there turn out to be many other situations in which similar bounds appear (see, e.g., Sec. 5 and [10]).

these ‘difference’ QWEIs bound the extent to which the energy density can drop below the vacuum expectation value.

A different approach to QWEIs was developed by Flanagan [16, 17] for massless scalar fields in two dimensions. The resulting bound is not only valid for a large class of averaging weights, but is also sharp. Yet another approach was initiated in work with Eveson [18] for averages along inertial trajectories in Minkowski space of dimension $d \geq 2$ using a large class of weight functions. For example, a scalar field of mass $m \geq 0$ obeys

$$\int \langle T_{00} \rangle_\psi(t, \mathbf{x}) g(t)^2 dt \geq -\frac{1}{16\pi^3} \int_m^\infty du |\hat{g}(u)|^2 u^4 Q_3(u/m) \quad (7)$$

in four dimensions, where $Q_3 : [1, \infty) \rightarrow \mathbb{R}^+$ is defined by

$$Q_3(x) = \left(1 - \frac{1}{x^2}\right)^{1/2} \left(1 - \frac{1}{2x^2}\right) - \frac{1}{2x^4} \ln(x + \sqrt{x^2 - 1}) \quad (8)$$

and obeys $0 \leq Q_3(x) \leq 1$ with $Q_3(x) \rightarrow 1$ as $x \rightarrow \infty$. In contrast to Flanagan’s bound, Eq. (7) is not sharp, and differs from it by a factor of $3/2$ in the $d = 2$, $m = 0$ case. Generalisations to static spacetimes [19], electromagnetism [20] and, on a slightly different tack, quantum optics [10] are known.

The following general QEI is based on Ref. [21] and essentially places the argument of [18] in a much more general setting. Consider a real, minimally coupled scalar field Φ of mass $m \geq 0$ propagating on a globally hyperbolic spacetime (M, g) . Each Hadamard state ω of the quantum field determines a two-point function

$$\omega_2(x, y) = \langle \Phi(x)\Phi(y) \rangle_\omega \quad (9)$$

which, in particular, satisfies the following properties:

- $\omega_2(\overline{F}, F) \geq 0$ for all test functions $F \in \mathcal{D}(M)$.
- $\omega_2(F, G) - \omega_2(G, F) = i\Delta(F, G)$ for all $F, G \in \mathcal{D}(M)$, where Δ is the advanced-minus-retarded fundamental solution to the Klein–Gordon equation. The important point is that the right-hand side is state-independent.
- The wave-front set [22] of ω_2 is constrained by $\text{WF}(\omega_2) \subset \mathcal{N}^+ \times \mathcal{N}^-$, where \mathcal{N}^\pm is the bundle of null covectors on M directed to the future (+) or past (−). This is the **microlocal spectrum condition**, which encodes the Hadamard condition [23]. All Hadamard two-point functions are equal, modulo smooth terms.

Given a second Hadamard state $\omega^{(0)}$, which we adopt as a reference state, the normal ordered two-point function

$$:\omega_2:(x, y) = \omega_2(x, y) - \omega_2^{(0)}(x, y) \quad (10)$$

is therefore smooth and symmetric and obeys

$$:\omega_2:(\overline{F}, F) \geq -\omega_2^{(0)}(\overline{F}, F). \quad (11)$$

The diagonal values $:\omega_2:(x, x)$ define the Wick square $\langle :\Phi^2: \rangle_\omega(x)$.

Now let g be a smooth, real-valued function, compactly supported in a single coordinate patch of (M, \mathbf{g}) , and define an averaged Wick square by

$$A(g, \omega) := \int \langle :\Phi^2: \rangle_\omega(x) g(x)^2. \quad (12)$$

Then, splitting the points in the definition of $:\Phi^2:$ by the introduction of a δ -function

$$A(g, \omega) = \int d\text{vol}(x) d\text{vol}(y) :\omega_2:(x, y) g(x) g(y) \delta_{\mathbf{g}}(x, y), \quad (13)$$

where $\delta_{\mathbf{g}}$ is the δ -function on (M, \mathbf{g}) . Passing to the coordinate chart containing the support of g , and writing the δ -function as a Fourier integral, we find

$$A(g, \omega) = \int \frac{d^4 k}{(2\pi)^4} \int d^4 x d^4 y :\omega_2:(x, y) g(x) g(y) (\rho(x) \rho(y))^{1/2} e^{-ik \cdot (x-y)}, \quad (14)$$

where, in these coordinates, $\rho(x) = |\det \mathbf{g}_{ab}(x)|^{1/2}$. Exploiting the symmetry of $:\omega_2:$, the k -integral may be restricted to the half-space with $k_0 > 0$ at the expense of a factor of 2. We then have

$$\begin{aligned} A(g, \omega) &= 2 \int_{k_0 > 0} \frac{d^4 k}{(2\pi)^4} :\omega_2:(\bar{g}_k, g_k) \\ &\geq -2 \int_{k_0 > 0} \frac{d^4 k}{(2\pi)^4} \omega_2^{(0)}(\bar{g}_k, g_k) \\ &\geq -2 \int_{k_0 > 0} \frac{d^4 k}{(2\pi)^4} \hat{F}(-k, k), \end{aligned} \quad (15)$$

where $g_k(x) = e^{ik \cdot x} g(x) / \rho(x)^{1/2}$ and $F(x, y) = g(x) g(y) (\rho(x) \rho(y))^{1/2} \omega_2^{(0)}(x, y)$. We may now invoke the microlocal spectrum condition and Prop. 8.1.3 in [22] to show that the right-hand side of the inequality is finite because the Fourier transform of F decays rapidly in the integration region. (We are using a nonstandard convention for the Fourier transform in which $\hat{f}(k) = \int dx f(x) e^{ik \cdot x}$.)

To convert this into a general quantum energy inequality, suppose f^{ab} is a tensor field for which, classically,

$$T_{ab} f^{ab} = \sum_{j=1}^N (P_j \phi)^2, \quad (16)$$

where P_j are partial differential operators with smooth, real, compactly supported coefficients. Then exactly the same argument yields a (finite) lower bound on $\int d\text{vol}(x) \langle :T_{ab}: \rangle_\omega(x) f^{ab}(x)$ simply by replacing ω_2 by $\sum_{j=1}^N (P_j \otimes P_j) \omega_2$ in the definition of F . Since the scalar field obeys the DEC and WEC precisely because the appropriately contracted stress tensor may be written in the ‘sum of squares’ form (16), our QEI has, as special cases, the quantum dominant/weak energy inequalities (QDEI/QWEIs).

Several remarks are appropriate here. First, the bound depends on the coordinate system chosen, so one has the freedom to sharpen the bound by modifying the coordinates.

Second, it is remarkable that the bound remains finite if the support of g (or f^{ab}) is shrunk to a timelike curve.³ The same is not true for averaging along null curves or within a spacelike slice, where one may show explicitly that the averaged quantity is unbounded from below [24, 25]. Third, the argument can be generalised to spin-one fields [26]. Fourth, restricted to static worldlines in static spacetimes, with the reference state chosen to be a static ground state, we find

$$\int dt \langle :T_{ab} u^a u^b: \rangle_{\omega}(\gamma(t)) g(t)^2 \geq - \int_0^\infty du Q(u) |\hat{g}(u)|^2, \quad (17)$$

where u^a is the four-velocity of the static worldline γ , and Q is monotone increasing and polynomially bounded.⁴ As a special case, we recover Eq. (7); bounds of the form Eq. (17) have also appeared in other contexts (see Sec. 5). Finally, a different approach to scalar field QEIs, which also employs microlocal techniques, can be found in [27].

One of the key ideas underlying the argument just given was the positivity of the classical expression $T_{ab} f^{ab}$. The situation is rather different in the case of a Dirac field, for which the classical [i.e., ‘first quantised’] energy density is, like the Hamiltonian, unbounded from both above and below. Positivity of the total energy emerges for the first time after renormalisation. For some time, this frustrated attempts to obtain a QWEI for spin- $\frac{1}{2}$ fields. The first success was due to Vollick [28], who adapted Flanagan’s proof [16] to treat massless Dirac fields in two dimensions. Subsequently, Verch and the present author used microlocal techniques to establish the existence of Dirac and Majorana QWEIs in general four-dimensional globally hyperbolic spacetimes [9]. However, the first explicit QWEI bound for Dirac fields in four dimensions has only been obtained very recently [29]. This bound is also of the form (17).

So far we have only discussed free quantum fields. The situation for general interacting fields is not yet clear (see the remarks below). However, it is known that all unitary, positive-energy conformal fields in two-dimensional Minkowski space obey QEIs [30] by an argument based on that used by Flanagan for massless scalar fields [16].

Finally, we should note that there are quantum field theories which do not satisfy QEIs. The simplest (and rather unphysical) example consists of an infinite number of fields with the same mass. More serious, perhaps, is the fact that the nonminimally coupled scalar field violates the energy conditions even at the classical level and is not expected to obey QEIs. In this regard, it is worth noting that the theory of Einstein gravity with a nonminimally coupled scalar field is mathematically equivalent⁵ (in the so-called ‘Einstein frame’) to the theory of a *minimally* coupled field plus gravity (see Ref. [31] for a review). In the Einstein frame, of course, QEIs do hold. It is possible that one may require a full theory of quantum gravity to assess the significance of the failure of QWEIs in the usual ‘Jordan frame’ (see [32] for a careful discussion of physics in different conformal frames). Olum and Graham [33] have also argued that interacting

³Indeed, the version of this argument in [21] considered only the case of averaging along a smooth timelike curve.

⁴If ω_0 is time-translationally invariant, but not a ground state, then $Q(u)$ has a tail in the negative half-line which decays rapidly as $u \rightarrow -\infty$.

⁵Equivalence holds provided the scalar field does not take Planckian values, a regime in which the nonminimally coupled theory is, in any case, unstable.

quantum fields can violate worldline QWEIs. They consider two coupled scalar fields, one of which is in a domain wall configuration; away from the wall, the second field experiences a static negative energy density (as often occurs near mirrors). This suggests strongly that the existence of QEIs for worldline averages is a special feature of the free field (or conformal fields in two dimensions [30]). However, it is still plausible that QEIs exist for spacetime averages of the stress-energy tensor. Consider a family of smearings whose spacetime ‘support radius’ is determined by a parameter λ . In the situation just described, sampling over longer timescales (say, by increasing λ) would also involve sampling over larger spatial scales, eventually meeting the (large) positive energy in the domain wall. It is certainly conceivable that the averaged energy density could still satisfy a lower bound which tends to zero as $\lambda \rightarrow \infty$ and diverges as $O(\lambda^{-4})$ as $\lambda \rightarrow 0^+$.

4 Connections with thermodynamics

Quantum inequalities originate from a 1978 paper of Ford entitled “Quantum coherence effects and the second law of thermodynamics” [12]. Ford argued that unconstrained negative energy fluxes (e.g., a superposition of right-moving modes with a left-directed flux) could be used to violate the second law of thermodynamics, by directing such a beam at a hot body to lower both its temperature and entropy. However, macroscopic violations of the second law cannot occur if the magnitude F and duration τ of the negative energy density flux are constrained by $|F| \lesssim \tau^{-2}$ because the absorbed energy would be less than the uncertainty of the energy of the body on the relevant timescale. This prompted Ford to seek mechanisms within quantum field theory which would limit negative energy fluxes and densities, and led ultimately to quantum inequalities of the type described in Sec. 3.

Recently, in work with Verch [34], a new twist has been added to the connection between quantum inequalities and thermodynamics: it turns out that there is a rigorous converse to Ford’s original argument. We consider quantum systems in static spacetimes of the form $\mathbb{R} \times \Sigma$ where the spatial section Σ is a compact Riemannian manifold. The algebra of observables, \mathfrak{A} is assumed to be a C^* -algebra on which the time translations $t \mapsto t + \tau$ are assumed to induce a strongly continuous one-parameter family of automorphisms α_τ , so that $(\mathfrak{A}, \alpha_\tau)$ is a C^* -dynamical system. We also assume that the system is endowed with an energy density $\rho(t, \mathbf{x})$ whose spatial integral over any hypersurface $\{t\} \times \Sigma$ generates the time evolution in the sense that

$$\int_{\Sigma} d\text{vol}_{\Sigma}(\mathbf{x}) \ell([\rho(t, \mathbf{x}), A]) = \frac{1}{i} \frac{d}{d\tau} \ell(\alpha_\tau(A)) \Big|_{\tau=0} \quad (18)$$

for sufficiently large classes of observables $A \in \mathfrak{A}$ and continuous linear functionals $\ell \in \mathfrak{A}^*$. (Precise definitions are given in [34].) One may now investigate the consequences of assuming that $\rho(t, \mathbf{x})$ satisfies various QWEI conditions, patterned on those obeyed by quantum fields. In particular, a state ω of the system is said to obey a **static quantum weak energy inequality** with respect to a class of states \mathcal{S} if, for each real-valued $g \in C_0^\infty(\mathbb{R})$ there is a locally integrable non-negative function $\Sigma \ni \mathbf{x} \mapsto q(g; \mathbf{x})$ such that

$$\int dt g(t)^2 [\langle \rho(t, \mathbf{x}) \rangle_\varphi - \langle \rho(t, \mathbf{x}) \rangle_\omega] \geq -q(g; \mathbf{x}) \quad (19)$$

for all $\varphi \in \mathcal{S}$ and $\mathbf{x} \in \Sigma$. The state ω is said to be **quiescent** if, in addition, each \mathbf{x} has an open neighbourhood U such that

$$\lambda \int_U d\text{vol}_\Sigma(\mathbf{x}) q(g_\lambda; \mathbf{x}) \longrightarrow 0 \quad \text{as } \lambda \rightarrow 0^+, \quad (20)$$

where $g_\lambda(t) = g(\lambda t)$. (One may regard this as a spatially averaged version of a difference AWEK.) On the assumption that \mathcal{S} is a sufficiently rich class of states, we proved, *inter alia*, the following result.

Theorem 1 *If a state $\omega \in \mathcal{S}$ obeys a static QWEI then the C^* -dynamical system admits a passive state. Moreover, if ω is quiescent then it is passive.*

We recall that the defining property of a **passive** state of a C^* -dynamical system is the impossibility of extracting net work from a system initially in such a state by a cyclical perturbation of the dynamics [35]. In this sense, the passivity criterion is an expression of the second law of thermodynamics; the force of the above result is that thermodynamic stability may be viewed as a consequence of QWEIs.

The abstract results of Ref. [34] are complemented by a detailed study of the free scalar field in static spacetimes with compact spatial sections. This does not immediately fit into our framework as the Weyl algebra describing the field theory is not a C^* -dynamical system. However, one may construct an auxiliary C^* -dynamical system to which the structural assumptions do apply. (Such complications would be absent for the Dirac field.)

These results lead to an interesting situation. As we have seen, QWEIs are consequences of the microlocal spectrum condition, while passivity is a consequence of QWEIs. Earlier work by Sahlmann and Verch [36] established that states of the scalar field obeying a certain passivity condition necessarily obey the microlocal spectrum condition. Thus the three conditions of passivity, QWEIs and the microlocal spectrum condition are mutually interconnected. And this is significant because these conditions may be interpreted as a stability conditions operating at different scales: microscopic [microlocal spectrum condition], mesoscopic [QWEIs] and macroscopic [passivity].

5 Quantum inequalities in quantum mechanics

A nice analogy to quantum energy inequalities may be found in the context of Weyl quantisation. In this procedure, a classical observable (i.e., a function on phase space) $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is represented in quantum mechanics by the operator F_w on $L^2(\mathbb{R}^n)$ with action

$$(F_w \psi)(x) = \int \frac{d^n y d^n p}{(2\pi\hbar)^n} F\left(\frac{x+y}{2}, p\right) e^{ip \cdot (x-y)/\hbar} \psi(y), \quad (21)$$

whose expectation values may be expressed in terms of the classical symbol $F(x, p)$ by

$$\langle F_w \rangle_\psi = \int \frac{d^n x d^n p}{(2\pi)^n} F(x, p) W_\psi(x, p), \quad (22)$$

where $W_\psi(x, p)$ is the Wigner function corresponding to ψ :

$$W_\psi(x, p) = \frac{1}{\|\psi\|^2} \int d^n y e^{ipy} \overline{\psi(x + \hbar y/2)} \psi(x - \hbar y/2), \quad (23)$$

As is well known, the Wigner function need not be everywhere positive, so it is clear that the positivity of F in no way entails the positivity of F_w . This mirrors the situation with energy density: even fields which obey the energy conditions classically will violate them in quantum field theory. Given sufficient regularity of the classical symbol F , however, the quantised observable F_w satisfies a sharp Gårding inequality [37] of the form

$$\langle F_w \rangle_\psi \geq -C(\hbar) \quad \forall \psi \in C_0^\infty(\mathbb{R}^n), \quad (24)$$

which, from our current standpoint, is precisely a quantum inequality. One may also investigate the specific example of energy densities in quantum mechanics. As in quantum field theory, the energy density at a point is unbounded from below, but time averages obey quantum inequalities of a form similar to Eq. (17) [38].

6 Physical Interpretation

Quantum energy inequalities demonstrate clearly that large negative energy densities and fluxes are associated with high frequencies (or short length-scales, as in the Casimir effect): averaging is required to obtain semibounded expectation values and it is crucial that the averaging function should decay sufficiently rapidly in the frequency domain in order that bounds of the form Eq. (17) are finite. Further insights have been provided by Ford and Roman [39], who discuss positive and negative energy densities in terms of the financial metaphor of credit and debt. Consider, for example, an energy density taking the form

$$\rho(t) = A\delta(t) + A(1 + \epsilon)\delta(t - T) \quad (25)$$

along some inertial worldline.⁶ Here, one can interpret A as the magnitude of ‘debt’ incurred, T as the term of the ‘loan’ and ϵ as the ‘interest rate’ due on repayment. Clearly a necessary condition for this to be the energy density of, say, a massless scalar field in two dimensions, is that it should satisfy

$$\int dt \rho(t) g(t)^2 \geq -\frac{1}{6\pi} \int dt |g'(t)|^2, \quad (26)$$

for all real-valued $g \in C_0^\infty(\mathbb{R})$, which is Flanagan’s QWEI [16]. Constraints on T and ϵ may be obtained in terms of A by substituting particular test functions g [39]. Sharper bounds are yielded [40] by rephrasing Eq. (26) as the condition that the differential operator

$$H_\rho = -\frac{d^2}{dt^2} + 6\pi\rho(t) \quad (27)$$

should be a positive quadratic form on $C_0^\infty(\mathbb{R})$. In the example given, it turns out that

$$T < \frac{1}{6\pi A} \quad \text{and} \quad \epsilon \geq \frac{6\pi AT}{1 - 6\pi AT}. \quad (28)$$

⁶This is to be regarded as a toy model for more realistic smooth energy distributions.

The two striking features are, firstly, that there is a maximum loan term and, secondly, that the interest rate is always positive and diverges as the maximum loan term is approached. Thus quantum fields act so as to restore net energy density positivity locally (rather than globally); negative energy densities are obtained only at the expense of a nearby positive energy density of greater magnitude. For further results in this direction see [41, 42].

One interesting consequence of the fleeting nature of negative energy densities is that it will be hard to observe them directly. Helfer [43] has argued, on the basis of various thought experiments, that quantum fields satisfy ‘operational energy conditions’: that is, the energy of any measurement device capable of resolving transient negative energy densities will necessarily be large enough that the net local energy density will be positive.

Finally, we mention two important applications of quantum energy inequalities. First, they have been used to place constraints on various “designer spacetimes” including warp drive models [44] and traversable wormholes [45] (see also [46]). Second, as already mentioned, Marecki has adapted quantum inequality arguments to bound fluctuations of the electric field strength in quantum optics [10]. It is a tantalising prospect that these results may have direct relevance to experiments in the near future.

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A Scaling limits

We briefly give some more details on the statement made at the end of Sec. 2. To do this we must briefly recall the notion of a **scaling limit**, introduced by Fredenhagen and Haag [7]. Our presentation is influenced by [47]. Consider a four-dimensional Lorentzian spacetime (M, \mathbf{g}) and fix a point $\bar{p} \in M$ and a chart neighbourhood $\kappa : U \rightarrow \kappa(U) \subset \mathbb{R}^4$ of \bar{p} . We assume that $\kappa(U)$ is convex and that $\kappa(\bar{p}) = 0$, and define a family of local diffeomorphisms σ_λ ($\lambda \in (0, 1]$) of U by

$$\kappa(\sigma_\lambda(p)) = \lambda \kappa(p). \quad (29)$$

Clearly these maps form a semigroup, with the properties $\sigma_\lambda \circ \sigma_{\lambda'} = \sigma_{\lambda\lambda'}$ and $\sigma_1 = \text{id}$, and contract U to the single point \bar{p} as $\lambda \rightarrow 0^+$. We also define an action on $\mathcal{D}(U^{\times n})$, i.e., test functions on the n -th Cartesian power of U , by

$$(\sigma_{\lambda*} f^{(n)})(p_1, \dots, p_n) = f^{(n)}(\sigma_\lambda^{-1}(p_1), \dots, \sigma_\lambda^{-1}(p_n)), \quad (30)$$

for $(p_1, \dots, p_n) \in \kappa(U)^{\times n}$ with $\sigma_{\lambda*} f^{(n)}$ vanishing elsewhere. [Note that [7] employs maps which are diffeomorphisms of the full manifold, leading to a correspondingly more restrictive definition of scaling limit in what follows.]

Now let $\omega^{(n)}$ be the hierarchy of n -point functions for the $T(\cdot)$ studied in Sec. 2 (or any other field with the properties assumed of T), defined by

$$\omega^{(n)}(f_1 \otimes \dots \otimes f_n) = \langle \Omega | T(f_1) \dots T(f_n) \Omega \rangle, \quad (31)$$

and which we assume to be distributions, $\omega^{(n)} \in \mathcal{D}'(M^{\times n})$. We say that Ω has a scaling limit at \bar{p} for the field T , if there exists a monotone function $N : (0, 1] \rightarrow [0, \infty)$ such that the limits

$$\widehat{\omega}^{(n)}(f^{(n)}) = \lim_{\lambda \rightarrow 0^+} N(\lambda)^n \omega^{(n)}(\sigma_{\lambda*} f^{(n)}) \quad (32)$$

exist and are finite for all $n = 1, 2, 3, \dots$ and all $f^{(n)} \in C_0^\infty(U^{\times n})$, and at least one of the $\widehat{\omega}^{(n)}$ is nontrivial (i.e., not the zero distribution). As shown in [7], the $\widehat{\omega}^{(n)}$ are distributions on $U^{\times n}$ and the function $N(\lambda)$ is ‘almost a power’, in the sense that there exists α such that

$$\lim_{\lambda' \rightarrow 0} \frac{N(\lambda\lambda')}{N(\lambda')} = \lambda^\alpha \quad (33)$$

for all $\lambda \in (0, 1]$. The number $d = 4 + \alpha$ is the **canonical dimension** of T at \bar{p} . Although our construction made use of a particular chart, the existence of a scaling limit is coordinate-independent, as is the function N .

In what follows, we assume that the scaling limit exists at \bar{p} with strictly positive canonical dimension d , which entails $\lim_{\lambda \rightarrow 0^+} \lambda^4 N(\lambda) = 0$.⁷ We also assume that $\widehat{\omega}^{(2)}$ is nontrivial; one may show from this that there exists non-negative $f \in \mathcal{D}(U)$ such that $\widehat{\omega}^{(2)}(f \otimes f) > 0$.⁸ Setting $f_\lambda = \lambda^{-4} \sigma_{\lambda*} f$, and considering the limit $\lambda \rightarrow 0^+$, we note that the support of f_λ tends to $\{\bar{p}\}$ while $\int f_\lambda d\text{vol}_g$ tends to a constant, which we may normalise to unity. Thus $f_\lambda \rightarrow \delta_{\bar{p}}$ as $\lambda \rightarrow 0^+$.

To complete the argument, we define $\zeta_\lambda = \|T(f_\lambda)\Omega\|$ and $\eta_\lambda = \langle \Omega | T(f)^3 \Omega \rangle / (2\zeta_\lambda^2)$. Clearly

$$\lambda^8 N(\lambda)^2 \zeta_\lambda^2 \rightarrow \lim_{\lambda \rightarrow 0^+} N(\lambda)^2 \omega^{(2)}(\sigma_{\lambda*} f^{\otimes 2}) \neq 0 \quad (34)$$

as $\lambda \rightarrow 0^+$; hence, because $\lambda^4 N(\lambda) \rightarrow 0$, we have $\zeta_\lambda \rightarrow \infty$. On the other hand,

$$\frac{\eta_\lambda}{\zeta_\lambda} = \frac{N(\lambda)^3 \omega^{(3)}(\sigma_{\lambda*} f^{\otimes 3})}{(N(\lambda)^2 \omega^{(2)}(\sigma_{\lambda*} f^{\otimes 2}))^{3/2}} \quad (35)$$

and therefore tends to a finite (possibly zero) limit. Comparing with the discussion in Sec. 2, we see that

$$\lim_{\lambda \rightarrow 0^+} \inf_{\substack{\psi \in D \\ \|\psi\|=1}} \langle \psi | T(f_\lambda) \psi \rangle = -\infty \quad (36)$$

⁷To see this, choose $0 < \lambda < 1$ such that $2\lambda^d < 1$, let λ_0 be such that $N(\lambda\lambda') < 2\lambda^\alpha N(\lambda')$ for all $0 < \lambda' < \lambda_0$ and consider the sequence $\lambda_n = \lambda_0 \lambda^n$. Then it is easy to see that

$$0 \leq \lambda_n^4 N(\lambda_n) < \lambda_0^4 (2\lambda^d)^n N(\lambda_0) \rightarrow 0$$

as $n \rightarrow \infty$. If N is monotone increasing, we are done; failing which, N is monotone decreasing and we argue as follows. For any $\lambda' < \lambda_0$, we define n by $\lambda' \in [\lambda_{n+1}, \lambda_n)$ so $n \rightarrow \infty$ as $\lambda' \rightarrow 0^+$ and then note that

$$0 \leq \lambda'^4 N(\lambda') \leq \lambda_n^4 N(\lambda_{n+1}) \leq (2\lambda^d)^{n+1} (\lambda_0/\lambda)^4 N(\lambda_0) \rightarrow 0$$

as $\lambda' \rightarrow 0^+$.

⁸Suppose $\widehat{\omega}^{(2)} \neq 0$. Then we may find $f, g \in \mathcal{D}(U)$ with $\widehat{\omega}^{(2)}(\bar{f} \otimes g) \neq 0$ because finite linear combinations of such tensor products are dense in $\mathcal{D}(U \times U)$. A polarisation argument, using the fact that $\widehat{\omega}^{(2)}$ is manifestly positive type, enables us to find f such that $\widehat{\omega}^{(2)}(\bar{f} \otimes f) > 0$, and we may take f real-valued without loss [by taking real and imaginary parts and applying Cauchy–Schwarz]. We split f into positive and negative parts, mollify to regain smoothness and apply the Cauchy–Schwarz argument again to obtain the required statement. Continuity of $\widehat{\omega}^{(2)}$ in the test functions is also used.

while $f_\lambda \rightarrow \delta_{\bar{p}}$. Thus the energy density at \bar{p} is unbounded from below, as claimed.

Finally, we remark that this result continues to hold even if the vacuum expectation value $\langle \Omega | T(\cdot) \Omega \rangle$ is nonvanishing, provided it is continuous at \bar{p} and $\tilde{T}(f) = T(f) - \langle \Omega | T(f) \Omega \rangle \mathbf{1}$ has a scaling limit of the required type, because the overall energy density is merely shifted by the finite constant $\langle \Omega | T(\bar{p}) \Omega \rangle$.

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